Relaxing partition admissibility in Cluster-DAGs: A causal calculus with arbitrary variable clustering

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Too Long; Didn't Read

We extend causal reasoning to arbitrary partitions of the variables in a causal graph.

Cluster-DAG Framework

In many domains, the full causal diagram is unknown. **Cluster DAGs (C-DAGs)** provide a **coarser abstraction**:

- Variables are grouped into clusters.
- Only inter-cluster relations are shown.
- Each C-DAG represents a collection of compatible graphs.

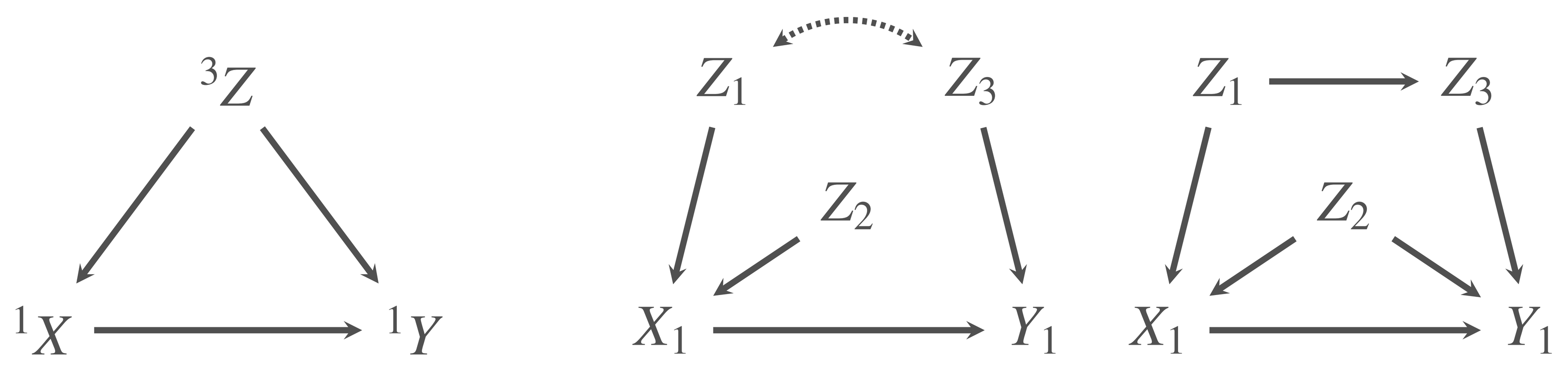

Figure 1. A C-DAG (left) and two compatible graphs (middle and right).

Key result. The original calculus is shown to be sound and complete for causal effect identification in the original setting.

Limitation. The partition had to be admissible, i.e., the resulting C-DAG had to be acyclic.

Our contribution. We remove the acyclicity constraint, allowing reasoning with any partition of variables.

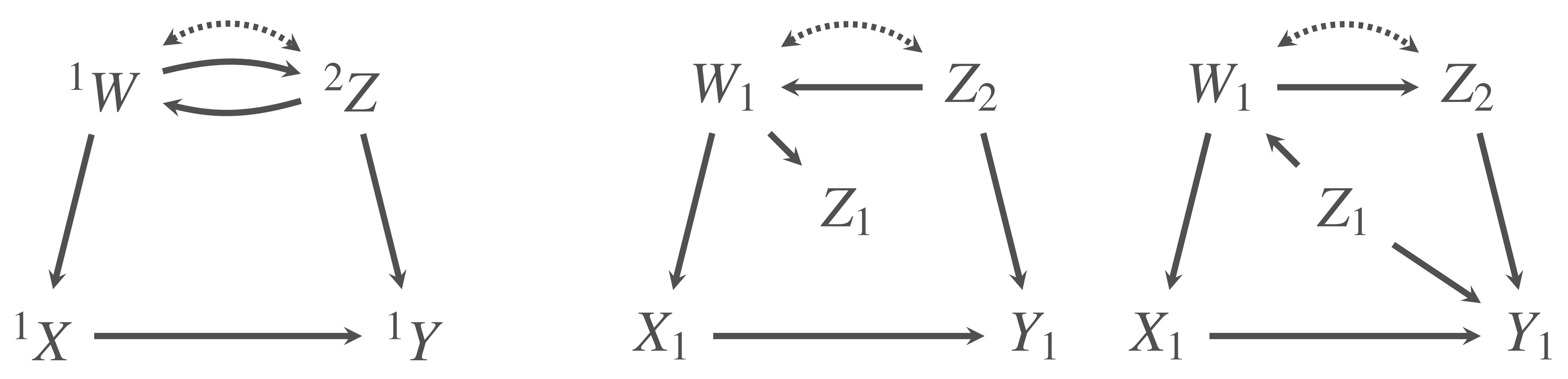

Figure 2. A cyclic C-DAG (left) and two compatible graphs (middle and right).

D-separation via Structures of Interest

Definition. A structure of interest σ is an ADMG, with a single connected component, in which each node V satisfies the following property:

- V has at most one outgoing arrow, or,
- V has two outgoing arrows but no incoming arrow.

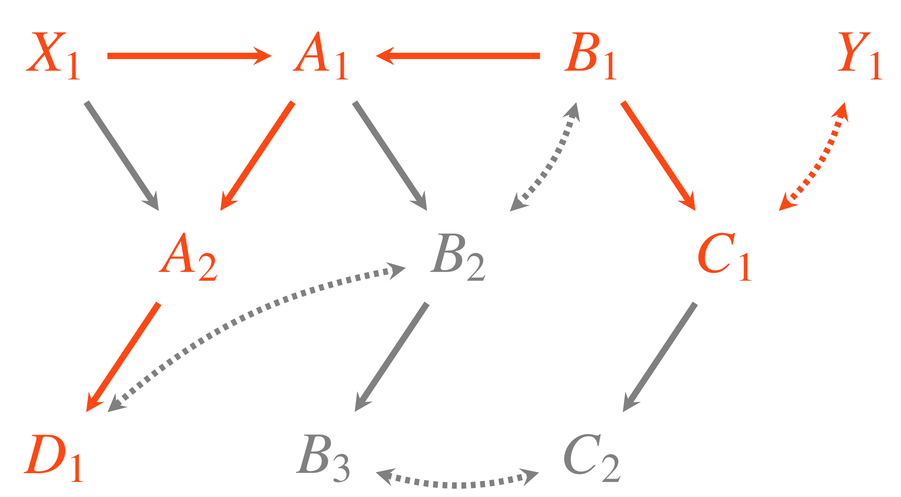

Figure 3. A causal graph containing a structure of interest in orange, that connects X_1 and Y_1 under $\{D_1, C_1\}$.

Theorem (D-connection with structures of interest). Let $\mathcal{G}$ be an ADMG. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be pairwise disjoint subsets of nodes of $\mathcal{G}$. The following properties are equivalent:

1. $\mathcal{X} \not\perp\!\!\!\perp_{\mathcal{G}} \mathcal{Y} \mid \mathcal{Z}$,
2. $\mathcal{G}$ contains a structure of interest σ such that $\mathcal{X} \not\perp\!\!\!\perp_{\sigma} \mathcal{Y} \mid \mathcal{Z}$.

Structures of interest:

- Encode all the information required for d-connection,
- Allow efficient reasoning over compatible ADMGs by avoiding an explicit path search.

Example of causal reasoning

In Figure 2, the following identities hold for all compatible graphs:

$$P(y_1 \mid \mathrm{do}(x_1)) = \sum_{w_1, z_1, z_2} P(y_1 \mid x_1, w_1, z_1, z_2) P(w_1, z_1, z_2)$$

$$P(x_1 \mid \mathrm{do}(w_1)) = P(x_1 \mid w_1)$$

A Sound and Complete Calculus

Key result. For each do-calculus rule, we can test whether it holds for all compatible graphs without having to enumerate them.

The calculus provides three rules (R1, R2, R3) that mirror Pearl's do-calculus:

- Each rule checks for the absence of certain structures of interest in modified versions of the C-DAG,
- If no such structure exists, the corresponding causal manipulation is valid,
- This avoids explicit enumeration of infinitely many compatible ADMGs.

Theorem. The calculus is sound and atomically complete for causal reasoning over C-DAGs.

Computational Efficiency

Reasoning over C-DAGs with large clusters can be computationally expensive.

Theorem (Infinity is at most 3). For any C-DAG $\mathcal{C}$ and any do-calculus rule $\mathcal{R}$, the following are equivalent:

(i) There exists an ADMG compatible with $\mathcal{C}$ in which $\mathcal{R}$ does not hold.
(ii) There exists such an ADMG, using only clusters of size at most 3, in which $\mathcal{R}$ does not hold.

Implication. Large clusters can be safely reduced to size 3, preserving causal validity while drastically simplifying computation.

This makes the calculus practically efficient even for complex abstractions.

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