Unveiling the Structure of Do-Calculus Reasoning via Derivation Graphs

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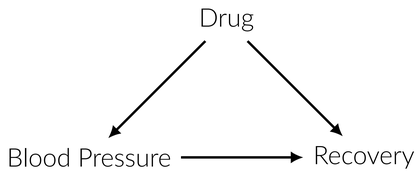
Too Long; Didn't Read

We introduce derivation graphs to explore the space of expressions equivalent under do-calculus and prove that any two such expressions are connected by at most four rule applications.

Introduction to Causal Reasoning

Causal Graphs.

- **Nodes:** Variables;
- **Edges:** Causal relationships:
  - $\rightarrow$: Direct Causation;
  - $\leftrightarrow$: Hidden common cause.



Interventions and Observations.

- **Observation:** I observe an event $\rightarrow$ conditioning
- **Intervention:** I *force* an event $\rightarrow$ do operator
- Example: $P(recovery \mid \mathrm{do}(drug), blood\ pressure)$: "Probability of recovery given the blood pressure and if the drug is administered."

Do-Calculus (Pearl (2009)). A set of three rewriting rules:

- Rule $\mathcal{R}_1$: insertion/deletion of observations
  $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W} \Rightarrow P(\mathbf{y} \mid \mathrm{do}(\mathbf{x}), \mathbf{z}, \mathbf{w}) \leftrightarrow P(\mathbf{y} \mid \mathrm{do}(\mathbf{x}), \mathbf{w})$,
- Rule $\mathcal{R}_2$: action/observation exchange
  $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}\underline{\mathbf{Z}}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W} \Rightarrow P(\mathbf{y} \mid \mathrm{do}(\mathbf{x}, \mathbf{z}), \mathbf{w}) \leftrightarrow P(\mathbf{y} \mid \mathrm{do}(\mathbf{x}), \mathbf{z}, \mathbf{w})$,
- Rule $\mathcal{R}_3$: insertion/deletion of actions
  $\mathbf{Y} \perp\!\!\!\perp_{\mathcal{G}_{\overline{\mathbf{X}}\overline{\mathbf{Z(W)}}}} \mathbf{Z} \mid \mathbf{X}, \mathbf{W} \Rightarrow P(\mathbf{y} \mid \mathrm{do}(\mathbf{x}, \mathbf{z}), \mathbf{w}) \leftrightarrow P(\mathbf{y} \mid \mathrm{do}(\mathbf{x}), \mathbf{w})$,

Example of Causal Reasoning

In the previous example, the following identities hold :

$$P(recovery \mid \mathrm{do}(bp)) = \sum_{drug} P(recovery \mid bp, drug) P(drug)$$

$$P(recovery \mid \mathrm{do}(drug)) = P(recovery \mid drug)$$

Why Another View on Do-Calculus?

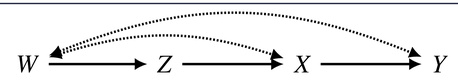


Figure 1. The Napkin Graph

Several derivations between $P(y \mid \mathrm{do}(w, x), z)$ and $P(y \mid \mathrm{do}(z), x)$ are valid:

$$P(y \mid \mathrm{do}(w, x), z) \rightarrow P(y \mid \mathrm{do}(w), x, z) \rightarrow P(y \mid \mathrm{do}(w, z), x) \rightarrow P(y \mid \mathrm{do}(z), x).$$

But others are not:

$$P(y \mid \mathrm{do}(w, x), z) \rightarrow P(y \mid \mathrm{do}(w), x, z) \rightarrow P(y \mid \mathrm{do}(w), x) \nrightarrow P(y \mid x) \nrightarrow P(y \mid \mathrm{do}(z), x).$$

How can we visualize all valid derivations for a given causal graph?
How can we resolve the choice of the order of application of do-calculus rules?

Derivation Graphs

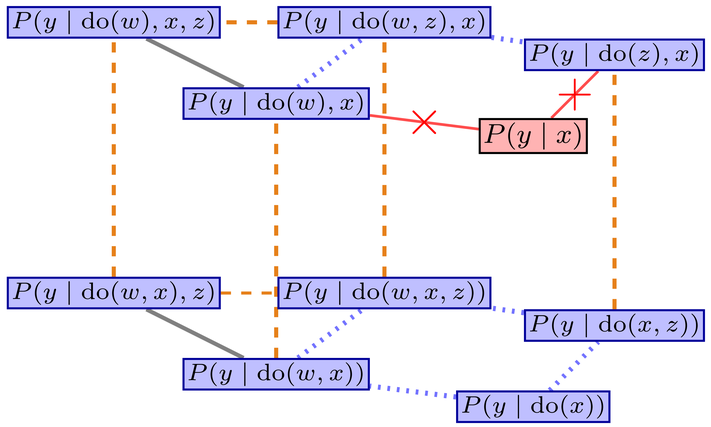


Figure 2. Connected component of $P(y \mid \mathrm{do}(x))$ in the derivation graph of the Napkin Graph. Edges represent atomic applications of the Do-Calculus rules: gray solid for $\mathcal{R}_1$, orange dashed for $\mathcal{R}_2$, and blue dotted for $\mathcal{R}_3$. Edges in red represent invalid derivations.

- **Nodes:** Every $P(y \mid do(x), z)$
- **Edges:** Valid applications of the Do-Calculus rules

It represents all valid derivations in a causal graph.

Main result: Simplifying Causal Reasoning

We study the redundancies and the commutativity of Do-Calculus:

- R1 is redundant: it can always be replaced by R2 and R3
- R2 and R3 commute partially

Every valid do-calculus derivation can be reordered into a canonical sequence with a fixed structure:

$$R_2^{\uparrow} \rightarrow R_3^{\uparrow} \rightarrow R_3^{\downarrow} \rightarrow R_2^{\downarrow}$$

$\uparrow$ = insertion | $\downarrow$ = deletion

Consequence. Equivalence between $P(\mathbf{y} \mid \mathrm{do}(\mathbf{x_1}), \mathbf{w_1})$ and $P(\mathbf{y} \mid \mathrm{do}(\mathbf{x_2}), \mathbf{w_2})$ can be decided using only four graphical validity checks.
Interpretation. The derivation can always be arranged to proceed in a single direction: *first we enrich the intervention set, then we simplify it back.*

Application: Deriving Multiple Estimators

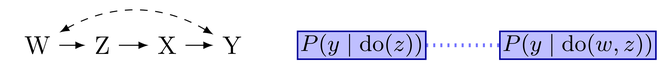


Figure 3. A causal Diagram and a Portion of its Derivation Graph.

Equivalent queries produce different estimators:

- $P(y \mid \mathrm{do}(z)) \rightarrow$ backdoor
- $P(y \mid \mathrm{do}(w, z)) \rightarrow$ frontdoor

Experiments:

- Synthetic Data
- Sachs Protein Signaling Dataset

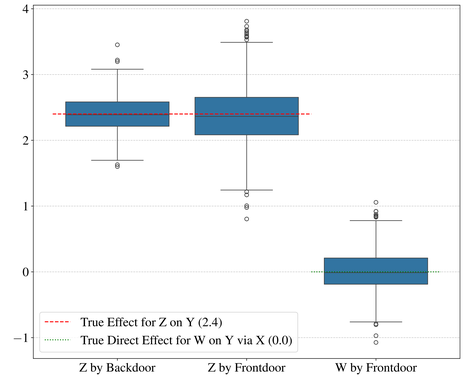


Figure 4. Variance Comparison

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